A First Course in Network Theory
Comparing Partitionings

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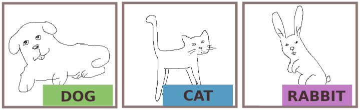
University of Strathclyde, Glasgow

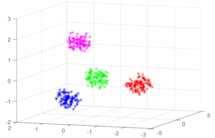
Evaluating Clustering Algorithms

Supervised Learning (Classification) VS Unsupervised Learning (Clustering)

Find the label of a data



Find groups of similar data



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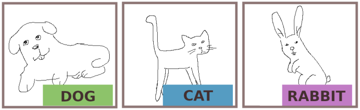

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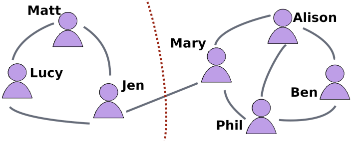

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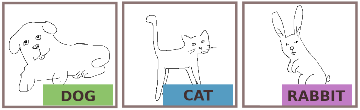

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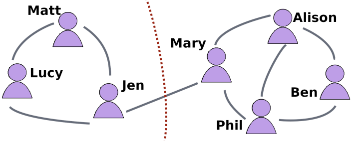

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Counting the errors (bad labels).

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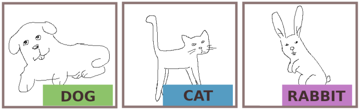

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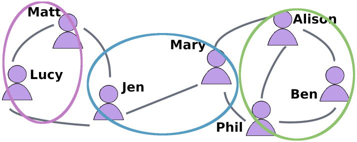

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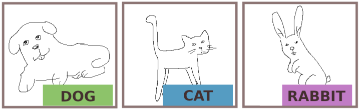

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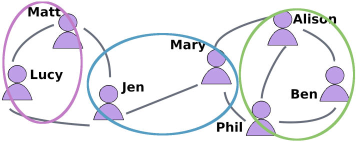

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$\mathcal{C} = \{C_1, ..., C_p\}$, $\mathcal{K} = \{K_1, ..., K_q\}$ are two partitionings on $\{1, ..., n\} = V$.

Agreement/Disagreement Table

$$\mathbf{N} = \begin{vmatrix} n_{1,1} & n_{1,0} \\ \\ n_{0,1} & n_{0,0} \end{vmatrix}$$

with

$$(TruePositives) \;\; n_{1,1} = \left| \left\{ (u,v) \in V \times V, u \neq v : \exists i, j \;\text{ with } u, v \in C_i \cap K_j \right\} \right|$$

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$$\text{Rand Index } RI(\mathcal{C}, \mathcal{K}) = \frac{n_{1,1} + n_{0,0}}{n_{1,1} + n_{0,0} + n_{1,0} + n_{0,1}}$$

Confusion Table

$$\mathbf{T} = \frac{1}{n}\begin{bmatrix} |C_1 \cap K_1| & \dots & |C_1 \cap K_q| \\ \vdots & \ddots & \vdots \\ |C_p \cap K_1| & \dots & |C_p \cap K_q| \end{bmatrix} \qquad \begin{cases} \sum_{j=1}^{q} \mathbf{T}(i,j) = \dfrac{|C_i|}{n} \\ \sum_{i=1}^{p} \mathbf{T}(i,j) = \dfrac{|K_j|}{n} \end{cases}$$

$$(1) \begin{cases} \text{Probability for a node } u \text{ to lie in a cluster } C_i \in \mathcal{C}: & Pr(u \in C_i) = |C_i|/n \\ \text{Probability for a node } u \text{ to lie in a cluster } K_j \in \mathcal{K}: & Pr(u \in K_j) = |K_j|/n \end{cases}$$

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Entropy $H(X)$ of a variable X is its uncertainty:

$$H(X) = -\sum_{x \in \mathcal{X}} Pr(X = x) \times log_2(Pr(X = x)).$$

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Mutual Info $MI(X, Y)$ is the reduction in X uncertainty due to knowing Y:

$$MI(X,Y) = H(X) - H(X|Y) = \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} Pr(x,y) log_2(\frac{Pr(x,y)}{Pr(x)Pr(y)})$$

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Using (1) and the confusion table $\mathbf{T}$:

$$MI(\mathcal{C}, \mathcal{K}) = \sum_{i=1}^{p} \sum_{j=1}^{q} \mathbf{T}(i,j) log_2 \left(\frac{n^2 \times \mathbf{T}(i,j)}{|C_i| \times |K_j|} \right)$$

Adjusted for Chance

An index $Ind(\mathcal{C}, \mathcal{K})$ can be **adjusted for chance**

$$AInd(\mathcal{C}, \mathcal{K}) = \frac{Ind(\mathcal{C}, \mathcal{K}) - \mathbb{E}[Ind(X, Y)]}{max(Ind(X, Y)) - \mathbb{E}[Ind(X, Y)]}$$

$\implies AInd(\mathcal{C}, \mathcal{K}) \approx 0$ when $\mathcal{C}$, $\mathcal{K}$ are independent.

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⚠ Requires to select random model.

✗ Not always easy to derive, and can be computationally awful.

$$E[\mathrm{MI}(U, V)] = \sum_{i=1}^{|U|} \sum_{j=1}^{|V|} \sum_{n_{ij}=(a_i+b_j-N)^+}^{\min(a_i, b_j)} \frac{n_{ij}}{N} \log \left(\frac{N.\, n_{ij}}{a_i b_j} \right) \frac{a_i! b_j! (N - a_i)! (N - b_j)!}{N! n_{ij}! (a_i - n_{ij})! (b_j - n_{ij})! (N - a_i - b_j + n_{ij})!}$$

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For the Rand Index, choice of the Permutation Model gives:

$$ARI(\mathcal{C}, \mathcal{K}) = \frac{2(n_{0,0}n_{1,1} - n_{0,1}n_{1,0})}{(n_{0,0} + n_{0,1})(n_{1,1} + n_{0,1}) + (n_{0,0} + n_{1,0})(n_{1,1} + n_{1,0})}$$