Reminder on Graphs & Basics on Networks

Complex Network Theory

Tutorial – 29/11/2021

1 Generalities on Graphs

Definition 1. Given $G = (V, E)$ some simple graph. Let us assume that $V = \{1, ..., n\}$. G is said to be

1. A **complete graph**, denoted by K_n, if it contains all possible edges.
2. A **cycle graph**, denoted by C_n, if its edges form a cycle connecting all nodes
3. A **path graph**, denoted by P_{n-1}, if it is a cycle graph missing one edge.
4. A **star graph**, denoted by $S_{1,n-1}$, if all nodes but one are solely connected to the same node.
5. A **wheel graph**, denoted by W_n, if it is the reunion of a star graph, and a cycle graph connecting nodes of degree 1.

Exercise 1. For each of the 5 above mentioned types of simple graphs:

- Draw it for $n = 5$.
- Explicit its adjacency matrix for $n = 5$.
- Explicit its number of edges according to n.
- Write its edge set.

Exercise 2. Let $\mathbf{I}_n \in \mathbb{R}^{n\times n}$ be the identity matrix, and $\mathbf{e}_n^i \in \mathbb{R}^n$ its ith column; $\mathbf{1}_n \in \mathbb{R}^n$ is the vector of all ones, and $\mathbf{J} = \mathbf{1}_n\mathbf{1}_n^T \in \mathbb{R}^{n\times n}$. Finally, let

$$\mathbf{N}_n^k = \begin{bmatrix} 0 & \dots & 1 & & \\ & \ddots & & \ddots & \\ & & \ddots & & 1 \\ & & & \ddots & \vdots \\ & & & & 0 \end{bmatrix}$$

be the matrix whose the kth upper diagonal is full of 1, and 0 elsewhere (stating that $\mathbf{I}_n = \mathbf{N}_n^0$).

Using the elements defined above, propose formulae to express the adjacency matrices of the 5 types of simple graphs from Definition 1.

Exercise 3. Which networks are isomorphic among those from Figure 1? Justify your answers.

Exercise 4. For all networks from Figure 2, are they (strongly) connected, weakly connected, or disconnected? Justify your answers.

Exercise 5. Show that the sum of the degrees of nodes in an anti-reflexive network is always an even number.

Exercise 6. The following property is a corollary of the property from Slide 10.

Property 1. Given $\mathbf{A} \in \mathbb{R}_+^{n\times n}$ the adjacency matrix of a graph, we have

$$a^k(i,j) \neq 0 \iff j \in \mathcal{N}_{out}^k(i) \iff i \in \mathcal{N}_{in}^k(j)$$

- Prove Property 1.
- Prove that it is not true for any $\mathbf{A} \in \mathbb{R}^{n\times n}$.
- Propose a solution to bypass this.

2 Markov Chains

Exercise 7. You are playing a head-and-tail game with one of your friends. That is, you toss a coin and each time the head shows up, your friend must give you £1. On the other hand, if it's a tail, you have to give £1 to your friend. Game is over once you or your friend has no more coin of £1. Here are a few details:

- The coin you are tossing has a probability q to show a tail once tossed.
- You start the game with x coins of £1, while your friend has initially y such coins.

What is the probability of you, respectively your friend, having $z\times$£1 after k tosses ? Propose a graph representation of the game that enables to answer this question via matrix-vector products.

Exercise 8. We have a set of web pages, that may point to some pages from the set. Some user is surfing on this environment, with the following behaviour:

- Being on a web page, the user can click a hyperlink to reach another page. The user has no favourite subject and no memory about the pages already visited, so the odds to click any hyperlink from the current web page are equal.
- The user can also chose to stop following hyperlinks, and rather reaches any page from the set. The probability that the user does this is α. Once again, when reaching a page from the set of web pages, the odds to reach any page are equal.

What is the probability for the user to be on a given page u after an infinite time surfing ? Use the adjacency matrix of the network representing web page interactions, and some of the elements defined in Exercise 2 to answer this.
Tips: You can start by investigating what happens when the user start from a specific web page, and then extending the result to any initial point by using Theorem 1.

Theorem 1. *Given* $\mathbf{A} \in {\mathbb{R}_+^*}^{n\times n}$*, and* $r(\mathbf{A})$ *its spectral radius. Thus,* $r(\mathbf{A})$ *is an eigenvalue of* $\mathbf{A}$ *whose corresponding eigenspace dimension is* 1*, and all other eigenvalues* λ *are such that* $|\lambda| < r(\mathbf{A})$*. Furthermore* $\exists \mathbf{x} \in {\mathbb{R}_+}^n$ *such that* $\|x\|_1 = 1$*, that verifies* $\mathbf{A}\mathbf{x} = r(\mathbf{A})\mathbf{x}$*.*

3 Bipartite Graphs

The chemical reaction of Bazsa and Lengyel can be described by the following equations:

$$H^+ + NO_3^- + HNO_2 \rightleftarrows 2NO_2 + H_2O$$

$$H^+ + [Fe]^{2+} + NO_2 \rightleftarrows [Fe]^{3+} + HNO_2$$

Exercise 9. Draw a unique network that represents both equations involved in the Bazsa and Lengyel's reaction as shown in the course slides. Since these reactions are reversible, we will assume the network to be undirected. What do you observe about the nodes ?

Exercise 10. Such a network is called a bipartite graph. Formally, a bipartite graph can be written

$$G = (U \cup V, E, \omega) \text{ with } \forall \{u, v\} \in E, u \in U \text{ and } v \in V$$

Prove that a bipartite graph contains no cycle[1] of odd length.

Exercise 11. A bipartite graph can be represented by a rectangular matrix $\mathbf{B} \in \Omega^{|U|\times|V|}$ such that

$$\mathbf{B}(u, v) = \begin{cases} \omega(\{u, v\}) & \text{if } \{u, v\} \in E, \\ 0 & \text{otherwise.} \end{cases} \quad (1)$$

For the graph of Exercise 8:

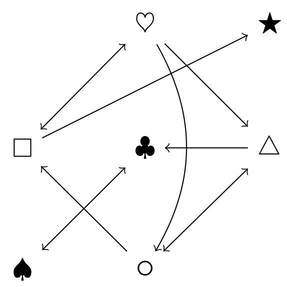
- Write its bipartite matrix.
- Write its adjacency matrix with a judicious choice of node ordering, and relate it to the bipartite matrix.
- Extend the result by expressing adjacency matrices of bipartite graphs using their bipartite matrices.

Exercise 12. The chemical reaction of the combustion of hydrogen within oxygen can be described using the following equations:

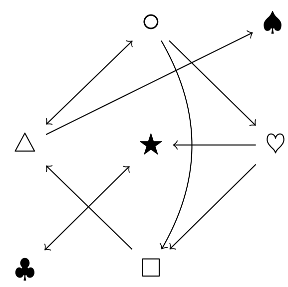
$$\begin{array}{rcl} H_2 + O_2 & \rightarrow & HO_2^\bullet + H^\bullet \\ H_2 + HO_2^\bullet & \rightarrow & HO^\bullet + HO_2 \\ H_2 + HO^\bullet & \rightarrow & H^\bullet + H_2O \\ H^\bullet + O_2 & \rightarrow & HO^\bullet + {}^\bullet O^\bullet \\ {}^\bullet O^\bullet + H_2 & \rightarrow & HO^\bullet + H^\bullet \end{array}$$

- Draw a unique directed network that represents the 5 equations of the combustion of hydrogen.
- Try to derive its bipartite matrix as stated in Eq 1. What is the problem ?
- Propose an adaptation of bipartite matrices from Eq 1 to this kind of networks.
- Investigate the extension of the result that relates bipartite matrices to adjacency matrices to your adapted definition of bipartite matrices.

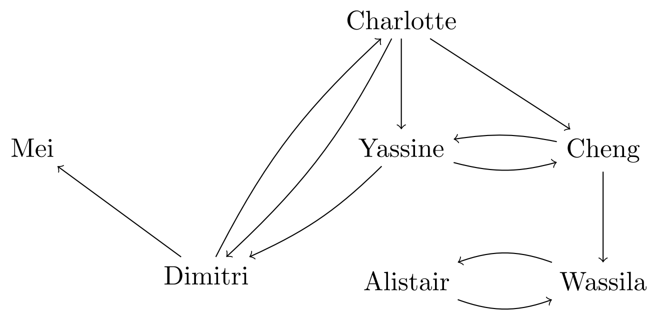
[1] A cycle is a path from a node to itself.



(a)



(b)



(c)

$$\begin{bmatrix} 0&1&1&1&0&0&0\\ 0&0&1&0&0&0&0\\ 1&0&0&0&0&1&0\\ 0&1&0&0&1&0&0\\ 0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0 \end{bmatrix}$$

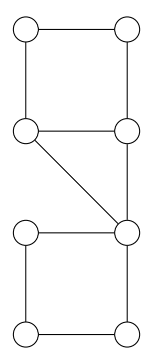
(d)

$$\begin{bmatrix} 1&0&0&0&0&0&0&0&0&0&0\\ -1&1&1&-1&0&0&0&0&0&0&0\\ 0&0&0&0&1&-1&0&0&0&0&0\\ 0&0&0&0&-1&1&1&0&0&0&0\\ 0&-1&0&1&0&0&0&-1&-1&0&0\\ 0&0&-1&0&0&0&0&1&0&1&-1\\ 0&0&0&0&0&0&-1&0&1&-1&1 \end{bmatrix}$$

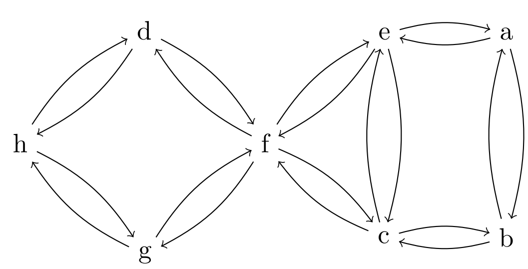
(e)

$$\begin{bmatrix} 0&0&0&0&0.33&0&0.33\\ 0&0&0&0&0&0&0.33\\ 0.33&0&0&0&0.33&0.33&0\\ 0&0&0&0&0&0&0\\ 0.33&0&0&0&0&0.33&0\\ 0&0&0.33&0.33&0&0&0\\ 0&0.33&0&0&0&0&0 \end{bmatrix}$$

(f)



(g)



(h)

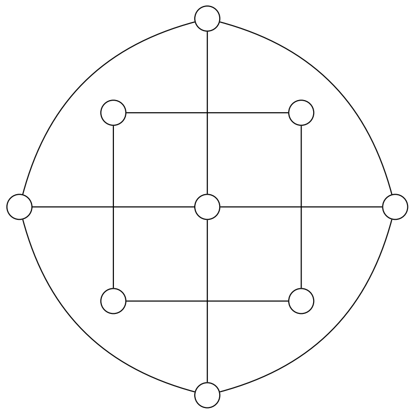
$$\begin{bmatrix} 0&1&0&0&1&0&0&0\\ 1&0&1&0&0&0&1&1\\ 0&1&0&1&0&0&1&0\\ 0&0&1&0&0&1&0&0\\ 1&0&0&0&0&0&0&1\\ 0&0&0&1&0&0&1&0\\ 0&1&1&0&0&1&0&0\\ 0&1&0&0&1&0&0&0 \end{bmatrix}$$

(i)

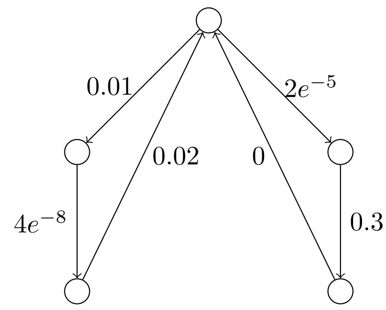
$$\begin{bmatrix} 0&0&0&0&1&0&1&0\\ 0&0&0&1&0&0&1&1\\ 0&0&0&1&0&0&0&1\\ 0&2&2&0&0&0&1&0\\ 2&0&0&0&0&1&0&0\\ 0&0&0&0&2&0&1&0\\ 2&2&0&2&0&2&0&0\\ 0&2&2&0&0&0&0&0 \end{bmatrix}$$

(j)

Figure 1



(a)



(b)

$$\begin{bmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
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\end{bmatrix}$$

(c)

$$\begin{bmatrix}
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1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
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1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}$$

(d)

$$\begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
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0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1
\end{bmatrix}$$

(e)

Figure 2