Homework 1 – Networks to Represent Differential Systems and Graph Laplacians

To return by 8am Monday 6 December 2021

1 One Dimension

In this exercise, we investigate the Laplacian of an application $u : [a, b] \to \mathbb{R}$ defined on the segment $[a, b]$ drawn in Figure 1.

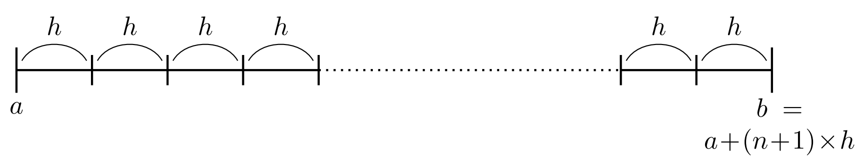

Figure 1: The segment $[a, b]$ regularly discretised with a discretising step h.

We furthermore assume that u is a twice differentiable function.

1. Recall how to get the formula of the centred approximation obtained by finite differences of the Laplacian of u at a point $x = a + k \times h$, with $k \in \{1, ..., n\}$. We will denote this approximation by $u_L^h(x)$ in the following.
2. Propose a representation as a **simple graph** of the discretised segment $[a, b]$ from Figure 1, and provide its adjacency matrix $\mathbf{A}$.
3. We denote by $\mathbf{u}, \mathbf{u}_L^h \in \mathbb{R}^n$ the two vectors such that:

$$\forall k \in \{1, ..., n\}, \begin{cases} \mathbf{u}(k) = u(a + k \times h) \\ \mathbf{u}_L^h(k) = u_L^h(a + k \times h) \end{cases}.$$

   Using basic linear algebra, derive a formula to express $\mathbf{u}_L^h$ involving $\mathbf{A}$, $\mathbf{D} = diag(\mathbf{A1})$, h and $\mathbf{u}$.

1.1 Hypothesis and Notations

- We can assume that $u(a) = u(b) = 0$.
- Given a vector $\mathbf{v} \in \mathbb{R}^n$, we denote by $diag(\mathbf{v})$ the diagonal matrix whose element (i, i) is equal to $\mathbf{v}(i)$.

2 Two Dimensions

We now investigate the Laplacian of an application $u : [a, b] \times [a, b] \mapsto \mathbb{R}$, defined on the square drawn in Figure 2.

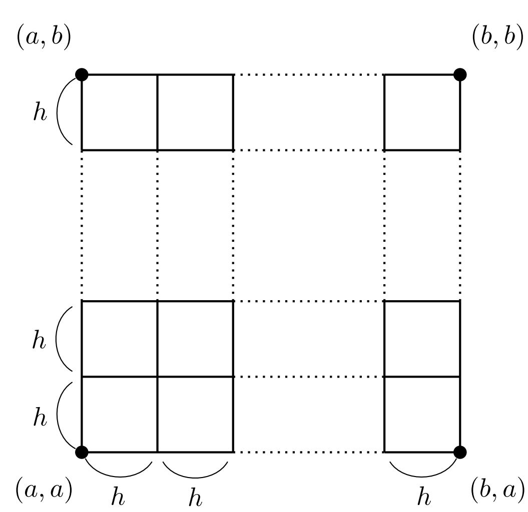

Figure 2: A square grid $(a, a) - (a, b) - (b, b) - (b, a)$, with $b = a + (n + 1) \times h$, discretised along x-axis and y-axis with a discretising step h.

As previously, u is assumed to be a twice differentiable function.

1. Recall how to get the formula of the centred approximation obtained by finite differences of the Laplacian of u at a point $(x, y) = (a + p \times h, a + q \times h)$, with $p, q \in \{1, ..., n\}$.
2. Propose a representation of the grid from the Figure 2 as a **simple graph**. Write its adjacency matrix $\mathbf{A}$ for $n = 3$. Write a Matlab function that returns the adjacency matrix of such a grid, provided n.
3. Similarly to what you have done for the one dimension problem, derive a matrix-vector-product formula to express the approximate Laplacian of u at points on the grid.

2.1 Hypothesis

- We can assume that $u(t, a) = u(a, t) = u(t, b) = u(b, t) = 0, \forall t \in [a, b]$.

3 Less Regular Grid

We finally investigate the Laplacian of a twice differentiable application $u : [a, b] \times [a, b] \mapsto \mathbb{R}$, defined on the rectangle drawn in Figure 2.

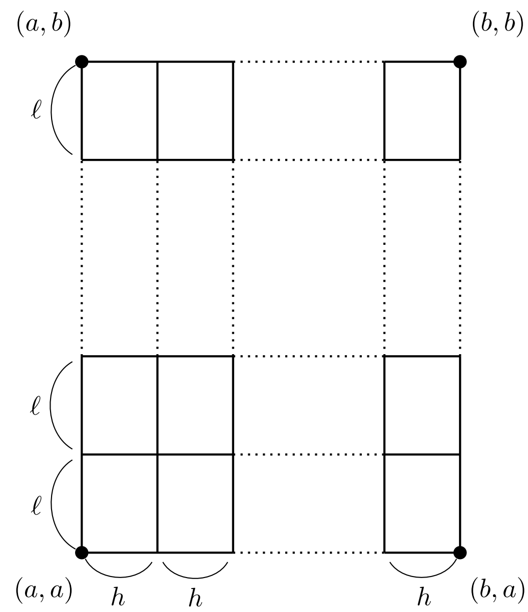

Figure 3: A grid $(a, a) - (a, b) - (b, b) - (b, a)$, with $b = a + (n + 1) \times h = a + (m + 1) \times \ell$, discretised along x-axis with a discretising step h, and along y-axis with a discretising step ℓ.

1. Provide the formula of the centred approximation obtained by finite differences of the Laplacian of u at a point $(x, y) = (a + p \times h, a + q \times \ell)$, with $(p, q) \in \{1, ..., n\} \times \{1, ..., m\}$.
2. How should we modify the graph derived for the regular grid to have a formula that resembles

$$\mathbf{u}_L^h = \frac{1}{h^2}(\mathbf{A} - \mathbf{D})\mathbf{u}?$$

3.1 Tips

- The graph that represents this grid is not simple.

4 Graph Laplacian

<u>Definition</u> Given a simple graph $G = (V, E)$, the matrix $\mathbf{L} = \mathbf{D} - \mathbf{A}$, with $\mathbf{D} = diag(\mathbf{A1})$, is called the Laplacian of G.

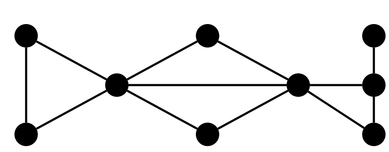

Figure 4: A simple graph.

Given the graph $H = (V_H, E_H)$ illustrated in Figure 4:

1. Write H's incidence matrix $\mathbf{B}_\mathcal{S}$ by considering that H is undirected. Then write H's incidence matrix $\mathbf{B}$ by considering that H is directed[1].

[1] These incidence matrices correspond to the *matrice d'incidence sommet-arête* and the *matrice d'incidence sommet-arc* you may have met in the *Théorie des Graphes* course.

2. Derive the two formulae that link the matrices $\mathbf{B}_{\mathcal{S}}\mathbf{B}_{\mathcal{S}}^T$ and $\mathbf{B}\mathbf{B}^T$ to the Laplacian of H. Prove these formulae for any simple graph.

3. Observe that, by assigning a random direction to H's edges, and denoting by $\mathbf{B}_{\mathcal{D}}$ the incidence matrix of the directed graph thus obtained, one gets $\mathbf{B}_{\mathcal{D}}\mathbf{B}_{\mathcal{D}}^T$ that equals H's Laplacian. Prove this result for any simple graph.

4. How would you extend the formula of Question 3. to undirected, anti-reflexive, but weighted graph ?