1)

$$A = \begin{bmatrix} 0&1&1&1&1&0&0&0&0&0 \\ 1&0&1&0&0&0&0&0&0&0 \\ 1&1&0&1&0&0&0&0&1&0 \\ 1&0&1&0&0&0&0&0&0&0 \\ 1&0&0&0&0&1&1&0&0&0 \\ 0&0&0&0&1&0&1&0&0&0 \\ 0&0&0&0&1&1&0&1&0&0 \\ 0&0&0&0&0&0&1&0&1&1 \\ 0&0&1&0&0&0&0&1&0&1 \\ 0&0&0&0&0&0&0&1&1&0 \end{bmatrix} , \quad D = \begin{bmatrix} 4 &&&&&&&&& \\ &2&&&&&&&& \\ &&4&&&&&&& \\ &&&2&&&&&& \\ &&&&3&&&&& \\ &&&&&2&&&& \\ &&&&&&3&&& \\ &&&&&&&3&& \\ &&&&&&&&3& \\ &&&&&&&&&2 \end{bmatrix}$$

$$\Rightarrow L = D - A = \begin{bmatrix} 4&-1&-1&-1&-1&\cdot&\cdot&\cdot&\cdot&\cdot \\ -1&2&-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ -1&-1&4&-1&\cdot&\cdot&\cdot&\cdot&-1&\cdot \\ -1&\cdot&-1&2&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ -1&\cdot&\cdot&\cdot&3&-1&-1&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&-1&2&-1&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&-1&-1&3&-1&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1&3&-1&-1 \\ \cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot&-1&3&-1 \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1&-1&2 \end{bmatrix}$$

2) a) v_1 is not the Fiedler Vector Becaux it is associated with 0 while the graph is connected : it is not the second smallest eigenvalue, but the smallest.

b)
$$L \times v_2 = \begin{bmatrix} -4-(-\sqrt{2}-1-\cancel{\sqrt{2}}+\cancel{\sqrt{2}}-1) = -2+\sqrt{2} = -1\times(2-\sqrt{2}) \\ -2\sqrt{2}-(-1-1) = -2\sqrt{2}+2 = -\sqrt{2}(2-\sqrt{2}) \\ -4-(-\sqrt{2}-1-\cancel{\sqrt{2}}+\cancel{\sqrt{2}}-1) = -1\times(2-\sqrt{2}) \\ -2\sqrt{2}+2 = -\sqrt{2}(2-\sqrt{2}) \\ 3(\sqrt{2}-1)-(\cancel{-1}+\cancel{1}+1) = 3\sqrt{2}-4 = (\sqrt{2}-1)(2-\sqrt{2}) \\ 2-(\sqrt{2}-\cancel{1}+\cancel{1}) = 1\times(2-\sqrt{2}) \\ 3-(\sqrt{2}-\cancel{1}+1+1) = 1\times(2-\sqrt{2}) \\ 3-(\cancel{1}+\sqrt{2}-\cancel{1}+1) = 1\times(2-\sqrt{2}) \\ 3(\sqrt{2}-1)-(\cancel{1}+1+1) = (\sqrt{2}-1)(2-\sqrt{2}) \\ 2-(\cancel{1}+\sqrt{2}-\cancel{1}) = 1\times(2-\sqrt{2}) \end{bmatrix} = (2-\sqrt{2}) \times v_2$$

Thus v_2 is an eigenvector of L associated with $2-\sqrt{2}$ (≈ 0.59)

c): $L v_3 = \left[\begin{array}{l} 4-\left(6/5+1+6/5+4/15\right) = 3-\left(12/5+4/15\right) = 3 - 40/15 = \frac{45-40}{15} = 1 \times 1/3 \\ 2 \times \frac{6}{5} - 2 = \frac{2}{5} = \frac{2}{5} \times \frac{3}{3} = \frac{6}{5} \times 1/3 \\ 2 \times \frac{6}{5} - 2 \qquad\qquad = \frac{6}{5} \times 1/3 \\ 4-\left(2 \times \frac{6}{5} + 1 - 4/15\right) = 3 - \frac{32}{15} \neq 1 \times 1/3 \\ \vdots \end{array}\right.$

Hence, v_3 is not an eigenvector of L (and can thus not be the Fiedler vector).

d): $L \times v_4 = \left[\begin{array}{l} 6 \times 4 - (\underbrace{-4+6-4-4}_{-6}) = 6 \times 4 + 6 = 5 \times 6 \\ \vdots \end{array}\right]$

If v_4 is an eigenvector, it is associated with $5 > 2-\sqrt{2}$ and this is thus not the Fiedler vector.

$\Rightarrow$ The Fiedler vector is then v_2.

3) Isoperimetric Ratio: our graph is $G=(V,E)$, $n = |V|$

Let $S \subset V$, thus $\varphi(S) = \frac{\mathrm{cut}(S)}{\min(|S|, n-|S|)}$.

4) $v_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \end{bmatrix}$, $(x_2, x_4, x_1, x_3, x_5, x_9, x_6, x_7, x_8, x_{10})$ is the list of v_2's coord. in ↑ order.

S	{2}	{2, 4}	{2, 4, 1}	{2, 4, 1, 3}	{2, 4, 1, 3, 5}	{2, 4, 1, 3, 5, 9}	{2, 4, 1, 3, 5, 9, 6}
$\varphi(S)$	2	$4/2 = 2$	$4/3 = 1.33$	$2/4 = 1/2$	$3/5 = 0.6$	$4/4 = 1$	$4/3$

S	{2, 4, 1, 3, 5, 9, 6, 7}	V \ {10}.
$\varphi(S)$	$3/2$	2

The minimum value of isoperimetric ratio obtained via the Sweep Cut Method is $1/2$, with $S = \{1, 2, 3, 4\}$.

The resulting partitioning is:

$$\mathcal{C} = \{\{1, 2, 3, 4\}, \{5, 6, 7, 8, 9, 10\}\}.$$

5) A partitioning S^* obtained via the sweep Cut Method always verifies the Cheeger inequalities:

$$\frac{\lambda_2}{2} \leq \varphi(S^*) \leq \sqrt{2 \lambda_2 d^{\omega}_{max}}$$

here: $\varphi(S^*) = 1/2$; $\frac{\lambda_2}{2} = \frac{2-\sqrt{2}}{2} = 1 - \frac{1}{\sqrt{2}} \approx 0.414 < 1/2 = \varphi(S^*)$

and

$$\sqrt{2 \lambda_2 d_{\omega}^{max}} = \sqrt{2 \times (2-\sqrt{2}) 4} = 2 \times \underbrace{\sqrt{2 \times \underbrace{(2-\sqrt{2})}_{> 1/2}}}_{>1} > 2 > \varphi(S^*).$$

6)

P & C:

Agreement / Disagreement

12 (n_{11})	9 (n_{10})	(21)
0 (n_{01})	24 (n_{00})	(24)
(12)	(33)	

Confusion Matrix

4	0
0	3
0	3

P, K

4	1	(5)
8	32	(40)
(12)	(33)	

2	2				4
		2	1		3
			1	2	3
2	2	2	2	2	

7) :

$$ARI(\mathcal{P}, \mathcal{E}) = \frac{2 \times (12 \times 24 - 0)}{24 \times 12 + 21 \times 33} \approx 0.5872$$

$$ARI(\mathcal{P}, K) = \frac{2 \times (4 \times 32 - 8)}{5 \times 33 + 12 \times 40} \approx 0.3721$$

$$MI(\mathcal{P}, \mathcal{E}) = \frac{4}{10} \times \log_2\left(\frac{10 \times 4}{16}\right) + \frac{3}{10} \times \log_2\left(\frac{10 \times 3}{18}\right) \times 2$$

≈ 0.971 (0.673 if log instead of $\log_2$).

$$MI(\mathcal{P}, K) = \frac{2}{10} \times \log_2\left(\frac{10 \times 2}{8}\right) \times 2 + \frac{2}{10} \times \log_2\left(\frac{10 \times 2}{6}\right) \times 2 + \frac{1}{10} \times \log_2\left(\frac{10 \times 1}{5}\right) \times 2$$

≈ 1.3710 (0.9503 if log instead of $\log_2$)

Thus, using ARI, $\mathcal{E}$ is considered closer to $\mathcal{P}$ than $\mathcal{K}$ while MI metrics, on the opposite, considers $\mathcal{K}$ as closer to $\mathcal{P}$ than $\mathcal{E}$.

This shows the inconsistency that may exist btw these measures.

8) $$Q(\mathcal{P}, G) = \frac{1}{14} \times \left(\left[5 - \frac{12^2}{56}\right] + 2\left[3 - \frac{8^2}{56}\right] \right) \approx 0.4688$$

$$Q(\mathcal{E}, G) = \frac{1}{14} \times \left(\left[5 - \frac{12^2}{56}\right] + 2 \cdot \left[7 - \frac{16^2}{56}\right] \right) \approx 0.347$$

$$Q(\mathcal{K}, G) = \frac{1}{14} \times \left(2 \cdot \left[1 - \frac{6^2}{56}\right] + 2 \cdot \left[1 - \frac{5^2}{56}\right] + \left[1 - \frac{6^2}{56}\right] \right) \approx 0.1556.$$

$$\phi(G, \mathcal{P}) = \frac{2}{12} + \frac{2}{8} + \frac{2}{8} = \frac{4+12}{24} = \frac{2}{3}$$

$$\phi(G, \mathcal{E}) = \frac{2}{12} + \frac{2}{16} = \frac{8+6}{48} = \frac{14}{48} = \frac{7}{24}$$

$$\phi(G, \kappa) = \frac{4}{6} + \frac{4}{6} + \frac{3}{5} + \frac{3}{5} + \frac{4}{6} = \frac{4}{2} + \frac{6}{5} = \frac{16}{5}$$

Modularity is more consistent than normalised Cuts here.