A First Course in Network Theory
Reminders and Basics about Graphs and Networks

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Generalities and Interest

Complex Networks are Graphs used to represent **complex systems**.

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Complex Systems *Systems composed of a* **large number of simple elements in interaction** *and exhibiting emerging phenomena*[1].

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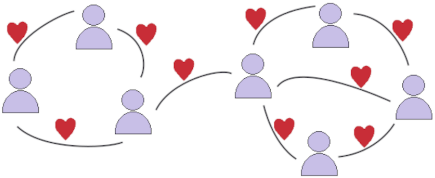

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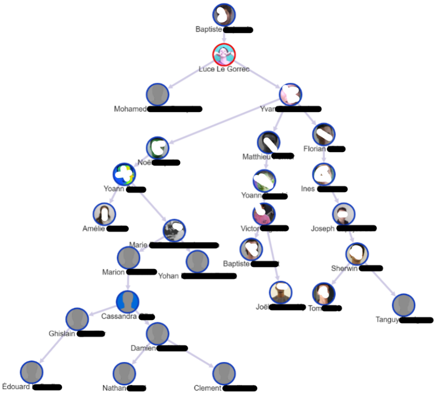

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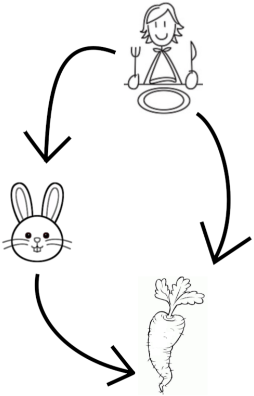

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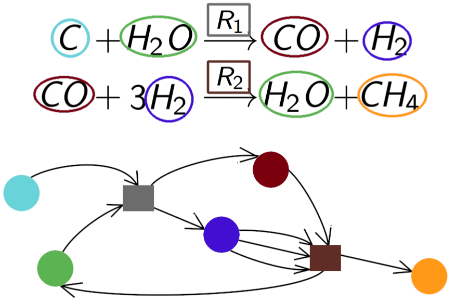

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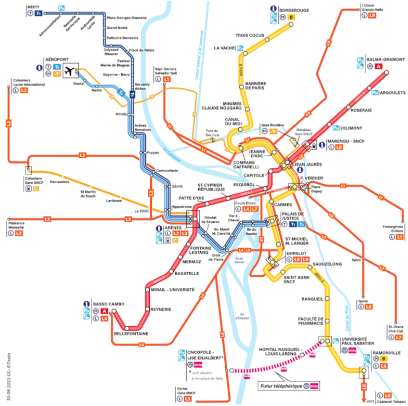

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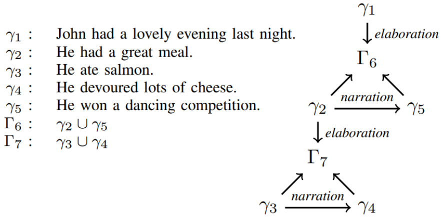

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Basics – Definitions

Definition A **Graph or Network** $G = (V, E, \omega)$ is a tuple of:

- a set V, called the vertex set (or node set). Elements of V are called **vertices or nodes**.
- a set $E \subset V \times V$, called the edge set (or link set). Elements in E are called **edges or links**.
- an application $\omega : E \to \Omega$, called the **edge weight function**.

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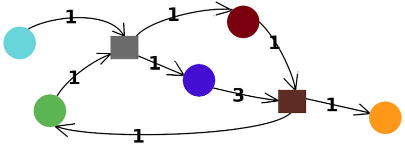

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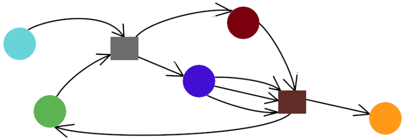

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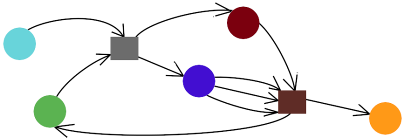

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Remark With this definition, information about nodes is lost.

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Definitions Given a graph $G = (V, E, \omega)$, and an edge $e \in E$:

- If $e(1) = e(2)$, then e is called a **(self-)loop**.

NB If $\forall e \in E$, e is **not** a self-loop, then G is called **anti-reflexive**.

- When $\omega : E \to \Omega = \{\omega_0\}$, then G is called an **unweighted graph** and ω may be omitted: $G = (V, E)$.

- If $\exists f \in E : \begin{cases} f(1) = e(2) \\ f(2) = e(1) \\ \omega(f) = \omega(e) \end{cases}$, then e is called a **bi-directed edge**.

NB If all edges are bi-directed, then G is said to be **undirected or non-directed**. One can choose to count each edge only once.

Definition A graph is called **simple** if it is i) non-directed; ii) unweighted; iii) anti-reflexive.

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Representing Graphs – Sets, Drawings and Matrices

Sets and Drawings

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Matrix representations We enforce $V = \{1, ..., n\}$ with $n = |V|$.

- **Incidence**: Given $G = (V, E)$ anti-reflexive and unweighted, with $m = |E = \{e_1, ..., e_m\}|$, the so-called **incidence matrix of** G is a matrix $\mathbf{B} \in \{-1, 0, 1\}^{n \times m}$ s.t. $b(i, k) = \begin{cases} -1 & \text{if } e_k(1) = i \\ 1 & \text{if } e_k(2) = i \\ 0 & \text{otherwise} \end{cases}$

NB For undirected graphs: two definitions.

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Question: How many adjacency matrices/incidence matrices possible for one graph ?

Representing Graphs – Isomorphisms

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- $|V_1| = |V_2|$.
- $|E_1| = |E_2|$.
- $\exists s : V_1 \to V_2$ a bijection s.t. $(i, j) \in E_1 \iff (s(i), s(j)) \in E_2$.
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Characterisation Two graphs of adjacency matrices respectively $\mathbf{A}_1$ and $\mathbf{A}_2$ are isomorphic iff $\mathbf{A}_1$ can be obtained from **simultaneous permutations of rows and columns** of $\mathbf{A}_2$.

Representing Graphs – Permutation matrices

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Exercise Prove that $\mathbf{P}^T = \mathbf{P}^{-1}$.

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Property Given $\sigma : \{1, ..., n\} \to \{1, ..., n\}$ a permutation. The matrix $\mathbf{P} \in \{0,1\}^{n \times n}$ s.t. $p(i,j) = 1 \iff j = \sigma(i)$ is a permutation matrix, and

$$\forall \mathbf{M} \in \mathbb{R}^{n \times n} \begin{cases} \mathbf{PM} = \mathbf{M}([\sigma(1), ..., \sigma(n)], :) \\ \mathbf{MP}^T = \mathbf{M}(:, [\sigma(1), ..., \sigma(n)]) \end{cases}$$

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Connectivity – Neighbourhood and degrees

Definition Given a graph $G = (V, E, \omega)$ and a node $v \in V$, we call:

- The **out-neighbourhood of** v the set $\mathcal{N}_{out}(v) = \{u : (v, u) \in E\}$,
  Its cardinal is called **the out-degree of** v: $d_{out}(v) = |\mathcal{N}_{out}(v)|$.
- The **in-neighbourhood of** v the set $\mathcal{N}_{in}(v) = \{u : (u, v) \in E\}$.
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Exercise Given $G = (V, E)$ an unweighted graph, express the in- and out-degrees of a node $v \in V$ using G's adjacency matrix, then G's incidence matrix.

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Connectivity – Interlude on weighted degrees

Definition Given a graph $G = (V, E, \omega)$ and a node $v \in V$, thus:

- The **weighted out-degree of** v is

$$d^{\omega}_{out}(v) = \sum_{u:(v,u)\in E} \omega((v, u)).$$

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- If G is **non-directed**, the **weighted degree of** v is

$$d^{\omega}(v) = \sum_{u:\{u,v\}\in E} \omega((u, v)).$$

Exercise Explain the limitation of weighted degree to non-directed graph. Any idea for extending this notion to directed graphs?

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Property Given $\mathbf{A} \in \{0,1\}^{n \times n}$ the adjacency matrix of an unweighted graph, **the value of $a^k(i,j)$ is the number of k-paths** from node i to node j.
Proof Exercise.

Connectivity – Irreducibility

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Characterisation **Irreducible matrices** are adjacency matrices of **strongly connected graphs**.
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