A First Course in Network Theory
Community Detection: a Partial Overview

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Community detection VS (Spectral) Graph Partitioning

- (Spectral) Graph Partitioning: For a specific application: parallel or distributed computations, building VLSI, etc.

$\Longrightarrow$ Known target, driven by the application.

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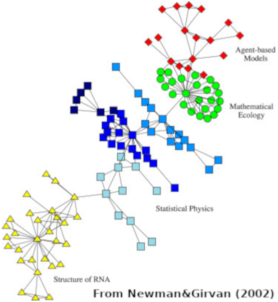

From Newman&Girvan (2002)

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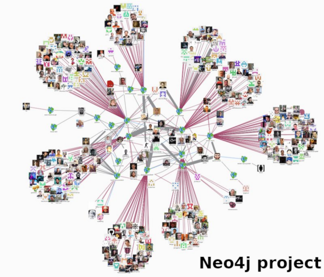

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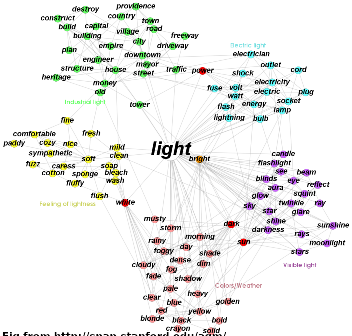

Fig from http://snap.stanford.edu/agm/

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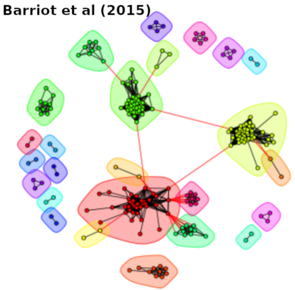

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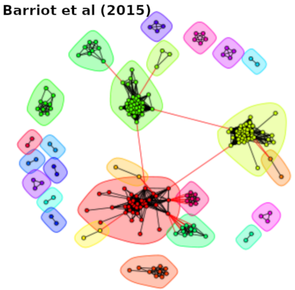

$\Longrightarrow$ A community $\approx$ a group of densely connected nodes, loosely connected with the rest of the network.

NB In the following, $G = (V, E, \omega)$ is an undirected graph. Community structure is a partitioning of V denoted $\mathcal{C} = \{C_1, ..., C_k\}$.

Attempts for formal definitions (Raddichi et al. 2004)

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- S is a **k-core** of G, with $k \in \mathbb{N}$, if

$$\forall u \in S, |\mathcal{N}(u) \cap S| \geq k$$

- S is a α-**clique** of G, with $\alpha \in]0, 1]$, if

$$2 \times |E \cap S \times S| / (|S| \times (|S| - 1)) \geq \alpha$$

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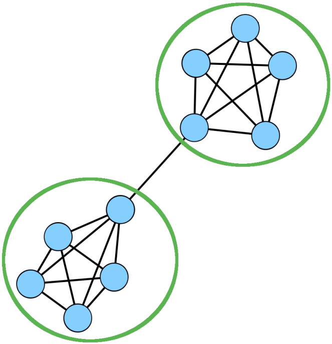

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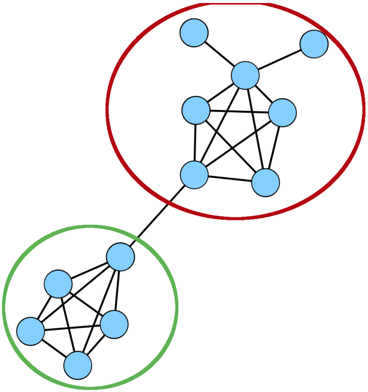

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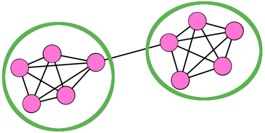

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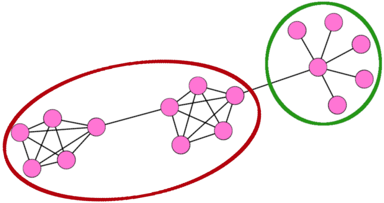

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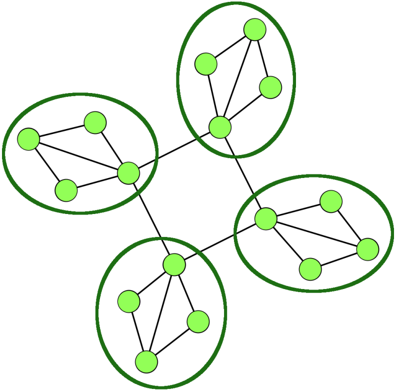

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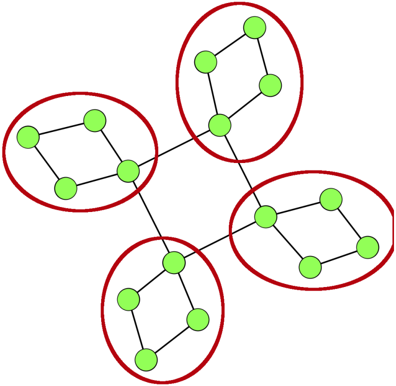

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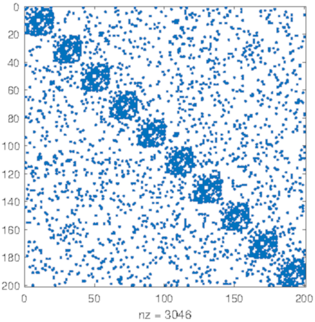

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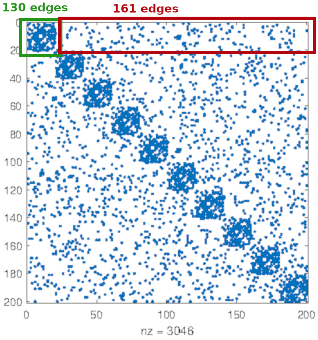

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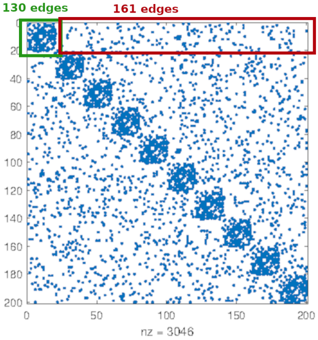

$\Longrightarrow$ No definitive definition, more a thumb rule.

Unformal definitions – Screenshot from (Veldt2019)

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"Graph clustering is the task of grouping the vertices of the graph into clusters taking into consideration the edge structure of the graph in such a way that there should be many edges within each cluster and relatively few between the clusters." [Schaeffer, 2007]

"One mesoscopic structure, called a community, consists of a group of nodes that are relatively densely connected to each other but sparsely connected to other dense groups in the network." [Porter et al., 2009]

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$\Longrightarrow$ $Mass(i) = d^\omega(i)$ makes the upper bound not degree-dependant.

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Modularity Given $m = |E|$ and assuming that $Pr(i \sim j) = \frac{d(i)d(j)}{2m}$ in G_R, then

$$\mathcal{Q}(G, \mathcal{C}) = \frac{1}{m} \sum_{C \in \mathcal{C}} |E \cap C \times C| - \mathbb{E}[|E_r \cap C \times C|] \quad (1)$$

$$= \frac{1}{m} \sum_{C \in \mathcal{C}} |E \cap C \times C| - \frac{vol(C)^2}{4m}. \quad (2)$$

Exercise Prove that (1) $\iff$ (2).

More on Modularity

Property Given, $\mathbf{A} \in \{0,1\}^{n \times n}$ the adjacency matrix of G, and stating $m = \sum_i d(i)/2$, one can write

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Property For a unweighted graph G,

$$-1/2 \leq \mathcal{Q}(G,C) \leq 1.$$

Proof Exercise for $\mathcal{Q}(G,C) \leq 1$.

A Bunch of Algorithms

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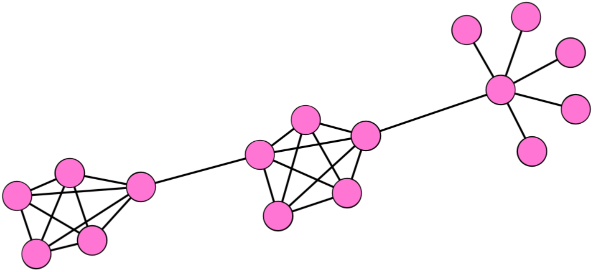
Idea Edges that **bridge communities** are involved in many **shortest paths**.
Cutting those edges should reveal the modular structure of the network.



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$$b(e) = \sum_{u \neq v \in V} \frac{\#\text{shortest paths between } u \text{ and } v \text{ that contain } e}{\#\text{shortest paths between } u \text{ and } v}.$$

Examples

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Idea Edges that **bridge communities** are involved in many **shortest paths**. Cutting those edges should reveal the modular structure of the network.

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$$b(e) = \sum_{u \neq v \in V} \frac{\#\text{shortest paths between } u \text{ and } v \text{ that contain } e}{\#\text{shortest paths between } u \text{ and } v}.$$

Examples

Algorithm 1) Compute the betweeness of each edge. 2) Remove the one with highest betweeness. 3) Update the betweeness of affected edges. 4) Go to Step 2.

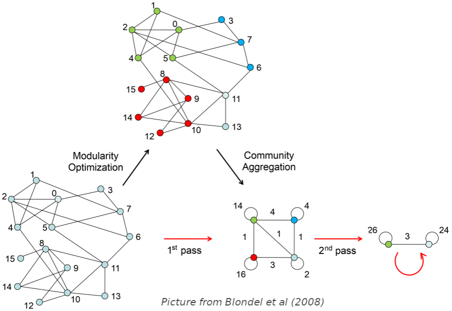
$\Longrightarrow$ A **divisive** algorithm that produces a **dendrogram**.

Example

A Bunch of Algorithms

Louvain (Blondel et al.2008)

Idea An efficient “heuristic” to maximise the Modularity $\mathcal{Q}(G,.)$[1].



Picture from Blondel et al (2008)

$\implies$ An **agglomerative** clustering.

✓ Efficient, accurate, used to maximise other measures (not efficiently for all).

[1] whose actual maximisation is NP complete

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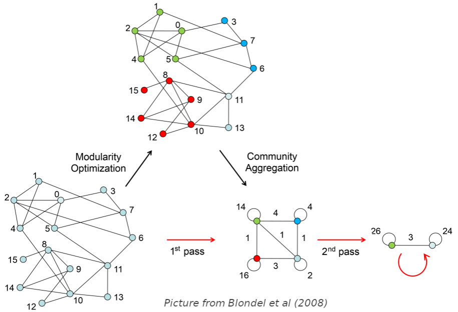

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$\Longrightarrow$ An **agglomerative** clustering.

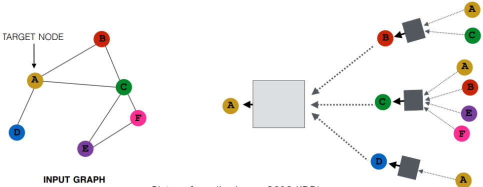
- ✓ Efficient, accurate, used to maximise other measures (not efficiently for all).
- ✗ A community returned by Louvain can be disconnected !
- Still one of the most used algorithms to date.

[1] whose actual maximisation is NP complete

A Bunch of Algorithms

Graph Convolutional Networks (Kipf&Welling2017)

Idea The state of a nodes depends on its neighbours: Convolutional neural networks on graphs.



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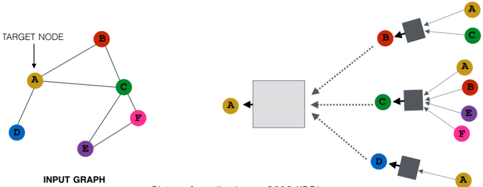

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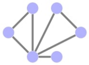
Graph Convolutional Layer $\sigma(\mathbf{A}\mathbf{H}^{(t)}\mathbf{W}^{(t)})$, with σ nonlinear function and

- $\mathbf{A} \in \mathbb{R}^{n \times n}$ the adjacency matrix,
- $\mathbf{H}^{(t)} \in \mathbb{R}^{n \times d_t}$ the "features" of nodes at layer t,
- $\mathbf{W}^{(t)} \in \mathbb{R}^{d_t \times d_{t+1}}$ the weights to learn in the tth layer.



$\rightarrow$ $\mathbf{H}^{(1)} = \sigma(\mathbf{M}\mathbf{X}\mathbf{W}^{(0)})$ $\rightarrow$ $\mathbf{H}^{(2)} = \sigma(\mathbf{M}\mathbf{H}^{(1)}\mathbf{W}^{(1)})$ $\rightarrow \cdots \rightarrow$ $\mathbf{Z} = \sigma(\mathbf{M}\mathbf{H}^{(k)}\mathbf{W}^{(k)})$

$\mathbf{M} \in \mathbb{R}^{n \times n}$ graph. struct.
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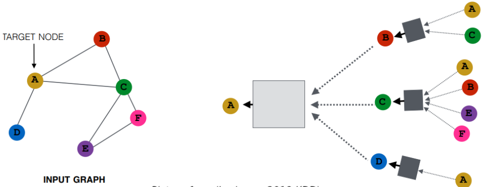

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Graph Convolutional Layer $\sigma(\mathbf{M}\mathbf{H}^{(t)}\mathbf{W}^{(t)})$, with σ nonlinear function and

- $\mathbf{M} = \widehat{\mathbf{D}}^{-1/2}\widehat{\mathbf{A}}\widehat{\mathbf{D}}^{-1/2}$, with $\widehat{\mathbf{A}} = \mathbf{I} + \mathbf{A}$ and $\widehat{\mathbf{D}} = diag(\widehat{\mathbf{A}}\mathbf{1})$
- $\mathbf{H}^{(t)} \in \mathbb{R}^{n \times d_t}$ the "features" of nodes at layer t,
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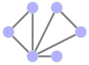 → $\mathbf{H}^{(1)} = \sigma(\mathbf{M}\mathbf{X}\mathbf{W}^{(0)})$ → $\mathbf{H}^{(2)} = \sigma(\mathbf{M}\mathbf{H}^{(1)}\mathbf{W}^{(1)})$ → $\cdots$ → $\mathbf{Z} = \sigma(\mathbf{M}\mathbf{H}^{(k)}\mathbf{W}^{(k)})$

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How close to the groundtruth is a community structure?

Problem Are two partitionings $\mathcal{C} = \{C_1, ..., C_p\}$ and $\mathcal{K} = \{K_1, ..., K_q\}$ close?

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<u>Rand Index</u> Counting the ratio of agreements:

$$RI = \frac{n_{1,1} + n_{0,0}}{n_{1,1} + n_{0,1} + n_{1,0} + n_{0,0}}, \text{ with}$$

		According to $\mathcal{C}$	
	$\#\{u, v\}$	$=$ community	$\neq$ communities
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Adjusted against chance Assuming a random clustering with number of elts/cluster:

$$ARI = \frac{RI - \mathbb{E}[RI]}{max(RI) - \mathbb{E}[RI]} = \frac{2(n_{0,0}n_{1,1} - n_{0,1}n_{1,0})}{(n_{0,0} + n_{0,1})(n_{1,1} + n_{0,1}) + (n_{0,0} + n_{1,0})(n_{1,1} + n_{1,0})}$$

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Definition: Given a rv X taking values $x_1, ..., x_k$, the **entropy** of X is

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Remark Entropy is said to be the **average level of surprise** of X. Why?

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Remark The **normalized MI** is most often used. An **adjusted MI** exists but less used because of its complexity.

Example

Conclusion

- **Community detection** means **finding consistent groups of nodes** within a network.
- What a good community should look like is highly **application dependant** (groups of densely connected nodes, but how?).
- This field has been **built on the fly** to answer real world problem of practitioners.
- **Modularity, Louvain algorithm, GCNs**, etc. are used because **they work globally well**, even if **they have flaws**.
- **No consensus on what does it means to be close for clusterings**.

? And for more complex networks ?